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SYSTEMS

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Abstract

Regulation in countries that have adopted defined contribution (DC) pension systems based on savings accounts typically includes minimum return guarantees (MRG) provisions to limit the risk of financial downturns. This paper studies the consequences of this regulation over asset allocation within a standard model of dynamic portfolio selection, where managers act strategically while making their investment decisions as in (Basak and Makarov, 2008, Strategic Asset Allocation with Relative Performance Concerns. Working Paper. London Business School).

We study a standard dynamic portfolio choice problem in a setting that includes two new ingredients: strategic interaction among portfolio managers and the presence of a MRG. The (pure strategy Nash) equilibrium portfolios are provided in closed-form in the Black and Scholes setting. They are shown to be weighted averages of investment rules that are themselves optimal in scenarios that may become optimal once the uncertainty has resolved.

Our results also suggest that MRG rules that rely on index-based benchmark portfolios (as opposed to peer-group ones) may help to mitigate some of the problems that arise when portfolio managers are too prone to relative performance concerns (i.e., the selection of myopic portfolios).

JEL classification: D81; G11; G18; and H55.
Key words: DC pension system; performance constraints; portfolio selection; strategic interactions.

1   Introduction

Demographic change has triggered a wave of pension reform around the world. Several of the countries that have undergone this trend have moved from pay-as-you-go (PAYGO) to defined contributions (DC) pension systems based on individual savings accounts. Regulation in these younger pension systems typically includes minimum return guarantee provisions to prevent the risk of financial downturns,[1] yet the consequences of these regulations over the asset allocation of the pension system remain to be fully understood.[2]

This paper studies the portfolio choice problem of pension fund managers that operates in a competitive setting, and that are subject to minimum return guarantees (MRG) such as those in Latin America and Central Europe pension systems. In particular, the paper takes a standard dynamic portfolio choice problem [e.g., Merton (1969, 1971)] and study the effects of MRG in a setting where fund managers have relative performance concerns and dynamic portfolio selection is subject to strategic considerations, along the lines of (Basak and Makarov, 2008, hereafter, BM).

For the case of the Black and Scholes (1973) setting, we are able to derive the (pure strategy Nash) equilibrium portfolio policies in closed form. We address two special cases: one where the MRG is based on a peer-group benchmark portfolio, and another where the benchmark portfolio is index-based.[3] Our results show that in both cases the equilibrium portfolios take a simple form: they are weighted sum of investment rules that are themselves optimal for scenarios that may be realized once the uncertainty is fully revealed. In the case where only two managers populate the economy, the scenarios correspond to the cases where both, neither, or at least one manager ends up restricted by the MRG constraint.

As shown by BM, equilibrium investment rules become (static) myopic when relative performance concerns among portfolio managers reach their peak. We show that this is also the case when the investment opportunity set is stochastic (e.g., time-varying stock returns, interest rates, volatility, correlations, etc.). As long-term investors are expected to take the changes in the opportunity set into account (e.g., Samuelson (1979), Campbell and Viceira (1999, 2001, 2002), Campbell et al. (2003)), this result suggests that MRG type of regulation should be careful when the intensity of competition among fund managers —proxied by relative performance concerns— is relatively high. In this sense, our results suggest that the use of a index-based benchmark portfolios may help to

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[1] Minimum return guarantees are common in new DC pension systems like those in Latin America and Central Europe; see for instance Turner and Rajnes (2001).

[2] In particular, previous studies abstract altogether from strategic interaction among pension fund managers; e.g., Boulier et al. (2001), Tépla (2001), Jensen and Sørensen (2001), Deelstra et al. (2003). The paper by Walker (2006) touches on several of the issues addressed in this paper by means of a static model.

[3] An index-based benchmark portfolio is composed of financial indexes classified by asset class (e.g., SP500 index, FTSE100 index, etc.) in a way that reflects the investment objectives and risk preferences of the investor, while a peer-group one is set as a combination of the asset allocation selected by the pension fund management industry itself. In the latter case, the implicit performance requirement is dictated as a function of the industry performance; Blake and Timmerman (2002).

mitigate this problem.

Our paper is related to several strands of the literature. On the issue of portfolio selection, it follows the classical literature of Samuelson (1969) and Merton (1969, 1971), using the martingale approach developed by Cox and Huang (1989, 1991) and Karatzas et al. (1987). On the issue of performance constraints, the paper is related to the literature on portfolio insurance [see, e.g., (Basak, 1995, 2002, and the references therein), Tépla (2001), Jensen and Sørensen (2001), Deelstra et al. (2003)] and capital guarantees [see, e.g., El Karoui et al. (2005)]. On the issue of portfolio choice and benchmarking, it is related to the papers by Basak et al. (2006, 2007, 2008). Finally, on the issue of portfolio choice and relative performance concerns, the paper is related to the seminal work of BM, from where we borrow extensively.

The paper is organized as follows. Section 2 presents the model and the main results. Section 3 summarizes the main findings and briefly concludes. All derivations are collected in an appendix at the end of the document.

2 A model economy

The starting point of our analysis is the model developed by BM, that includes strategic considerations into an otherwise standard dynamic portfolio choice problem, on top of which we introduce a minimum return guarantee and study the consequences over the resulting (pure strategy Nash) equilibrium portfolios.

2.1 The economic setting

We consider a continuous-time finite horizon economy with a frictionless financial market comprised of two financial assets: a (locally) riskless bond or money market account (B), and a risky stock (S). The dynamics of the prices of these assets is governed by the following laws of motion:

$$dB_t/B_t = r_t dt \quad \text{and} \quad dS_t/S_t + \delta_t dt = \mu_t dt + \sigma_t dW_t,$$

where $\{(r_t, \delta_t, \mu_t, \sigma_t) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}/\{0\}\}_{0 \le t \le T}$ correspond to the (instantaneous) interest rate, dividend rate, expected return, and the volatility of the stock, respectively, while $W_t \in \mathbb{R}$ is a standard Brownian motion process. All stochastic processes are assumed to be well defined, in the sense that B and S have a strong solution, and the (Novikov) condition

$$\mathbb{E}_0 \left[\frac{1}{2} \int_0^T \theta_t^2 dt \right] < \infty$$

is assumed to be satisfied, where $\theta \equiv \sigma^{-1}(\mu - r)$ is the market price of risk (MPR).

The economy is populated by asset managers in charge of managing the savings of a mandatory

DC pension system. For the sake of simplicity we study an economy populated by two asset managers.[4] Each asset manager has to chose an investment policy, $\pi_i \equiv \{\pi_{it} : \Omega \times [0, T] \mapsto \mathbb{R}\}_{0 \leq t \leq T}$, where π_i denotes the fraction of the pension fund that manager $i \in \{1, 2\}$ invests in the risky asset over time, given a fund with initial value of $x_i > 0$ at time 0. Formally, the dynamics of the pension fund of manager i is governed by the following law of motion:

$$\begin{cases} dX_{it}^{\pi}/X_{it}^{\pi} = (1 - \pi_{it}) \, r_t dt + \pi_{it} \, (dS_t/S_t + \delta_t dt) \, ; \\ X_{i0} = x_i > 0; \quad X_{it}^{\pi} \geq 0, \, \forall t \in [0, T]; \quad X_{iT}^{\pi} \equiv X_{iT}. \end{cases} \tag{1}$$

We consider an economy in which asset managers have absolute and relative performance concerns, i.e., they care about the terminal value of the fund under management, but also about their performance relative to that of their peers. In particular, we adopt the *envy* interpretation advanced by BM and define the preferences of each asset manager as a function of final wealth and its relative performance:[5]

$$v_i(X_{iT}, \hat{R}_{iT}) \equiv \frac{\left(X_{iT}^{1-\beta_i} \hat{R}_{iT}^{\beta_i} \right)^{1-\gamma_i}}{1 - \gamma_i}, \quad i \in \{1, 2\}, \tag{2}$$

where $\hat{R}_{iT} \equiv (X_{iT}/x_i) \div (X_{jT}/x_j)$ is the relative return between fund i and j, $\gamma_i > 0$ is the parameter of relative risk aversion, and $\beta_i \in [0, 1]$ is a parameter capturing the *relative performing bias*, i.e., the extent to which relative performance concerns alter the investment decisions of a normal manager.[6] In particular, when $\beta_i = 0$, we are back in the traditional case without strategic interaction, while the case $\beta_i = 1$ implies that the only concern of manager i is to beat manager $j \neq i$ in terms of $\hat{R}_{iT}$.

We introduce the MRG concerns of the asset manager as a benchmarking restriction to the portfolio problem [Basak et al. (2006, 2008)] of the form,

$$R_{iT} \geq R_T e^{-\varepsilon_i}, \quad i \in \{1, 2\}, \tag{3}$$

where $R_{iT} \equiv (X_{iT}/x_i)$, $R_T \equiv (X_{1T} + X_{2T})/(x_1 + x_2)$ is the peer-group average return, and $\varepsilon_i > 0$ is the allowed shortfall, i.e., $\varepsilon_i = \infty$ implies that the manager is unrestricted (as in BM), while $\varepsilon_i = 1\%$ means that the maximum log-return shortfall allowed is of 1% relative to the peer-group benchmark.

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[4]The analysis that follows can be extended for $N \geq 2$ along the lines of Remark 1 in BM.

[5]BM studied two possible interpretations for the function v_i in equation (2), *envy* and *fund flows*. Under the latter case, the combination of standard CRRA preferences and a specific (smooth) form of fund flows deliver equation (2), while the parameters β_i and γ_i are functionally related. In the case of the envy interpretation, on the other hand, β_i and γ_i are independent parameters.

[6]If the preferences of pension plan members are appropriately represented by $(1 - \gamma_i)^{-1}(X_{iT}^{1-\gamma_i})$, $\beta_i \neq 0$ can also be interpreted as a measure of the agency costs due to the activity of delegated portfolio management.

The reasoning behind this modeling choice is to think that the board of the asset management company deals with the MRG concern by imposing a relative performance or benchmarking constraint to the portfolio manager; Basak et al. (2006, 2007, 2008).

Remark 2.1 *Notice that in the current setting R_T is influenced by the relative size of each pension fund. In particular, the relative performance constraint in (3) is less stringent for the portfolio manager in charge of the bigger pension fund.*

2.2 Strategic interaction

Following BM we introduce strategic interaction among asset managers by relying on the martingale approach of Cox and Huang (1989, 1991) and Karatzas et al. (1987). In particular, this can be accomplished by taking (X_{1T}, X_{2T}) as the relevant *strategy space*, and restricting our analysis to the pure strategy Nash equilibrium concept solution.

We consider the case where managers decide over (X_{1T}, X_{2T}) simultaneously, similar to what happens in the classical Cournot duopoly game.

Definition 2.1 (Cournot best response function) *Given manager j's selection of X_{jT}, manager's i Cournot best response function is the solution to the following maximization problem*

$$\max_{X_{iT} \in \mathcal{A}} v_i(X_{iT}, \hat{R}_{iT}) \quad s.t. \quad \begin{cases} \mathbb{E}_0[\xi_T X_{iT}] \leq x_i \\ R_{iT} \geq R_T e^{-\varepsilon_i} \end{cases}, \tag{P0}$$

for $i, j \in \{1, 2\}$, $i \neq j$, where $\mathcal{A} \equiv \{X_{iT} \geq 0 : v_i < \infty\}$,

$$\xi_T \equiv \exp\left(-\int_0^T \left(r_t + \frac{1}{2}\theta_t^2\right) dt - \int_0^T \theta_t dW_t\right)$$

is the unique stochastic discount factor (SDF) that is compatible with the absence of arbitrage opportunities.

Proposition 2.1 *Manager's i Cournot best response function is given by*

$$\hat{X}_{iT} = \begin{cases} (y_i \xi_T)^{-1/\gamma_i} X_{jT}^{\beta_i(\gamma_i - 1)/\gamma_i} & \text{if } \xi_T \leq y_i^{-1} K_i^{-\gamma_i} X_{jT}^{-\bar{\gamma}_i} \\ K_i X_{jT} & \text{if } \xi_T > y_i^{-1} K_i^{-\gamma_i} X_{jT}^{-\bar{\gamma}_i} \end{cases}, \tag{4}$$

for $i, j \in \{1, 2\}$, $i \neq j$, where $\bar{\gamma}_i = \gamma_i(1 - \beta_i) + \beta_i$, $K_i = x_i/(e^{\varepsilon_i}(x_i + x_j) - x_i)$ and $y_i > 0$ is defined in Proposition A.1.

Proposition 2.1 shows that the Cournot best response function for manager i, $\hat{X}_{iT}$, has two parts. One that captures the behavior of the asset manager when unconstrained (i.e., the segment

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for low values of ξ_T, equivalent to equations (15) and (16) in BM), and a second one that captures the behavior when the asset manager is constrained by the benchmarking restriction (i.e., the segment for high values of ξ_T).

Corollary 2.1 (log investor is strategically myopic) *Notice that an unrestricted log utility manager is not only myopic in the classical sense (i.e., not considering the fluctuations in the investment opportunity set), but also it is myopic in the sense of not reacting to its rival's actions.*

Corollary 2.1 records an additional form of myopia for the case of the log investor (i.e., $\gamma_i = 1$), which in addition to not act in response to changes in the investment opportunity set, it neither does so in terms of the opponent strategy.

We also notice that the expression in (4) has a similar structure to those in Proposition 2 in BM. In our case though, the specific form of the best response function arises due to the benchmarking constraint, as opposed to the preferences under analysis. In this sense, our problem is closer to the literature on portfolio insurance, and European capital guarantees; Basak (1995, 2002), (Karatzas and Shreve, 1998, Chapter 3), El Karoui et al. (2005).

Corollary 2.2 (competition and limits of relative performance concerns) *Suppose that the economy is as in section 2.1. Then, when both managers are unrestricted ($\varepsilon_1 = \varepsilon_2 = \infty$) and the relative performing bias of both managers is maximum ($\beta_1 = \beta_2 = 1$), the optimal investment policy of each manager converges to the myopic solution $\pi_{it}^* = (\mu_t - r_t)/\sigma_t^2$, $i \in \{1,2\}$.*

Corollary 2.2 extends de findings in BM for the case of a stochastic investment opportunity set. The result implies that the limiting behavior of the relative performance bias is the myopic portfolio, which should not come as a surprise, as the myopic portfolio is known to be the one that maximizes the expected growth of terminal wealth (e.g., Merton (1990), Bajeux-Besnainou and Portait (1997), Platen (2005)). Still, what is a bit surprising is the fact that the limiting behavior of relative performing concerns —easily associated with a more competitive environment— may result in equilibrium portfolios that are likely to be suboptimal for long-term investors, such as pension funds (e.g., Samuelson (1979), Campbell and Viceira (1999, 2001, 2002), Campbell et al. (2003)). We believe this allows new foundation for optimal contracting (principal-agent) problems within asset management activities, as the framework where the trade-off is between effort and output seems to be less appropriate to the task.

2.3 Equilibrium portfolios under benchmarking constraints

Similar to the case with asymmetric relative performance concerns studied by in section 4 of BM, the presence of a benchmarking constraint may also prevent equilibrium to exist. Equilibrium is

shown to take the form of a weighted sum,

$$X^*_{iT} 1_{\{\xi_T \in \Xi_{uu}\}} + X^*_{uT} 1_{\{\xi_T \in \Xi_{ux}\}} + X^*_{xT} 1_{\{\xi_T \in \Xi_{xu}\}} \tag{5}$$

where X^*_{iT} stand for the optimal wealth of manager $i \in \{1,2\}$ when neither manager end up restricted by the benchmarking constraint at time T, X^*_{uT} and X^*_{xT} stand for the optimal wealth when either manager $j \neq i$ or manager i is restricted by the benchmarking constraint, respectively, while the sets Ξ_{uu}, Ξ_{ux} and Ξ_{xu} denote the domain of ξ_T for which each of the previous cases are valid. Existence, uniqueness and multiplicity of equilibria in this context requires the union, union and intersection, and intersection of these sets to be, respectively, nonempty, the positive real line and the empty set, and nonempty. Further details are provided in Proposition A.1 and Corollary A.1 in section A.1 for an economy as the one presented in section 2.1.

2.3.1 Peer-group benchmarking

We now move to the characterization of the equilibrium portfolios. To this end, we record first a well known result that connects our problem with the one studied by BM and help us to develop the necessary notation before stating our main result.

Corollary 2.3 (Merton (1971)) *When manager $i \in \{1,2\}$ is unrestricted ($\varepsilon_i = \infty$) and the relative performing bias is absent ($\beta_i = 0$), the optimal investment policy is given by*

$$\pi^M_{it} = \frac{\mu_t - r_t}{\gamma_i \sigma_t^2} + \pi^H_{it}$$

where π^H_{it} is the hedging component of the optimal portfolio identified in the proof. Moreover, when (r, σ, μ) are constant $\pi^H_{it} = 0$ and $\pi^M_{it} = \pi^M_i$.

Proposition 2.2 (equilibrium portfolio under peer-group benchmarking) *Suppose that (r, σ, μ) are constant and that the parameters of the model are such that*

$$1 - \bar{\gamma}_1 \left(\beta_2 \left(\gamma_2 - 1\right) + \gamma_1\right)/\Gamma < 0, \quad 1 - \bar{\gamma}_2 \left(\beta_1 \left(\gamma_1 - 1\right) + \gamma_2\right)/\Gamma > 0, \quad 1 - \bar{\gamma}_2/\bar{\gamma}_1 < 0,$$

where $\bar{\gamma}_i = \gamma_i + (1 - \gamma_i)\beta_i$, and $\Gamma = \gamma_1 \gamma_2 (1 - \beta_1 \beta_2) + \beta_1 \beta_2 (\gamma_1 + \gamma_2 - 1)$. Then the equilibrium portfolio policy of manager $i \in \{1,2\}$, $i \neq j$, is given by

$$
\begin{aligned}
\pi^*_{it} = {} & f^*_{i1t} \left[\frac{\gamma_i}{\gamma(\beta_i, \beta_j)} \pi^M_i + \frac{\phi_i\left(\hat{g}_{i2}(\xi_t)\right) - \phi_i\left(\hat{g}_{i1}(\xi_t)\right)}{\left[\Phi_i\left(\hat{g}_{1i}(\xi_t)\right) - \Phi_i\left(\hat{g}_{2i}(\xi_t)\right)\right]} h_t \right] \\
& + f^*_{i2t} \left[\frac{\gamma_i}{\bar{\gamma}_i} \pi^M_i + \frac{\phi(\hat{g}_{i3}(\xi_t))}{\left[1 - \Phi\left(\hat{g}_{i3}(\xi_t)\right)\right]} h_t \right] + f^*_{i3t} \left[\frac{\gamma_j}{\bar{\gamma}_j} \pi^M_j + \frac{-\phi\left(\hat{g}_{i4}(\xi_t)\right)}{\Phi_2\left(\hat{g}_{i4}(\xi_t)\right)} h_t \right]
\end{aligned}
\tag{6}
$$

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where $\gamma(\beta_i, \beta_j) \equiv (\gamma_i \gamma_j (1 - \beta_i \beta_j) + \beta_i \beta_j (\gamma_j + \gamma_i - 1))/(\gamma_j + \beta_i (\gamma_i - 1))$, $\Phi(\cdot)$ and $\phi(\cdot)$ stand for the c.d.f. and p.d.f. of the standard normal distribution, $h_t \equiv (\sigma \sqrt{T-t})^{-1}$, and $\{\{f^*_{ikt}\}_{1 \leq k \leq 3}, \{\hat{g}_{ik}\}_{1 \leq k \leq 4}\}$ are defined in the proof that is in the appendix.

Proposition 2.2 characterizes the optimal investment rule for the case where the MRG is based on a peer-group benchmark portfolio. The resulting expression corresponds to a weighted sum (as $f^*_{ikt} \in [0, 1]$ and $f^*_{i1t} + f^*_{i2t} + f^*_{i3t} = 1$) of portfolio policies that are themselves optimal for different scenarios that can occur at time T. The quantities attached to each weighting factor involve two components: the equilibrium portfolio policy that finances the equilibrium terminal wealth that is optimal in the corresponding scenario (see equation (5)), and the marginal change in its conditional probability of occurrence. For instance, in the case of the term attached to the first weight (f^*_{i1t}),

$$\frac{\gamma_i}{\gamma(\beta_i, \beta_j)} \pi^M_i + \frac{\phi(\hat{g}_{i2}(\xi_t)) - \phi(\hat{g}_{i1}(\xi_t))}{[\Phi(\hat{g}_{i1}(\xi_t)) - \Phi(\hat{g}_{i2}(\xi_t))]} h_t,$$

the term on the LHS stands for the optimal investment rule in the scenario where neither of the two portfolio managers is restricted by the MRG constraint at time T, while the term on the RHS corresponds to the marginal change at time t in the conditional probability that the respective scenario turns out to be optimal at time T. The term on the RHS is also present in cases of incentive schemes rendering local convexities in preferences (e.g., Basak et al. (2007, 2008), Castañeda (2006)). The statement in Corollary 2.2 also applies in this case, as $\gamma(1, 1) = 1$ and hence the term on the LHS boils down to the myopic portfolio.

The quantities attached to the other weighing factors follow a similar structure. In particular, the term on the LHS attached to the second and third factor corresponds to the optimal investment policy in the scenario where either manager j and i are restricted by the MRG constraint, respectively. In the case of the second term, the equilibrium portfolio shows the effects of the relative performance bias. When the bias is completely absent (i.e., $\beta_i = 0$ and hence $\bar{\gamma}_i = \gamma_i$), the equilibrium portfolio turns out to be the unconstraint optimal policy, π^M_i. On the contrary, when the bias reaches its peak (i.e., $\beta_i = 1$ and hence $\bar{\gamma}_i = 1$), the equilibrium portfolio corresponds to the myopic portfolio. The third term —associated with the case where manager i is restricted— follows the same logic. This time, though, it is the intensity of manager's j bias that matters. When the bias is completely absent (i.e., $\beta_j = 0$ and hence $\bar{\gamma}_j = \gamma_j$) manager i sticks to manager j's unconstraint portfolio policy, π^M_j. On the contrary, when the relative performing bias of manager j is at its maximum (i.e., $\beta_j = 1$ and hence $\bar{\gamma}_j = 1$), manager i adopts the optimal growth portfolio.

It is also worth noticing that the fraction invested in equity in the cases where at least one manager is constrained is affected by the value of the risk aversion parameter and the relative performing bias. In particular, when $\gamma_i > 1$ it follows that $\bar{\gamma}_i(\beta_i) : [0, 1] \mapsto [\gamma_i, 1]$ and hence $(\gamma_i/\bar{\gamma}_i) \pi^M_i \geq \pi^M_i$, while the opposite is true when $\gamma_i < 1$.

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2.3.2 Index-based benchmarking

In order to contrast our previous results, we consider the case where the benchmarking constraint is index based. To this end we introduce the index-based analog of the performance constraint (3), i.e.,

$$R_{iT} \geq R_{iT}^Y e^{-\varepsilon_i}, \qquad i \in \{1, 2\}, \tag{7}$$

where $R_{iT}^Y \equiv Y_{iT}/Y_{i0}$ and Y_{iT} as the terminal value of the dynamics embedded in the following system:

$$\begin{cases} dY_{it}^\pi/Y_{it}^\pi = \left(1 - \pi_t^Y\right) r dt + \psi_t \left(dS_t/S_t + \delta dt\right); \\ Y_{i0} = x_i > 0; \quad Y_{it}^\pi \geq 0, \, \forall t \in [0, T]; \quad Y_{iT}^\pi \equiv Y_{iT}; \end{cases} \tag{8}$$

where $\pi^Y \equiv \{\pi_t^Y \in \mathbb{R}\}_{0 \leq t \leq T}$ is the investment policy of the benchmark portfolio relevant to manager $i \in \{1, 2\}$, which is given by the fraction invested by the benchmark portfolio in the risky asset. For simplicity, we will assume that $\pi_t^Y = \pi^Y \in [0, 1]$, $\forall t \in [0, T]$.

Remark 2.2 *Notice that the only difference between Y_{1t}^π and Y_{2t}^π is the initial value of the benchmark portfolio, which is set equal to be value of the fund managed by each manager at time 0. Both managers hence face an identical performance constraint, i.e., $Y_{1t}^\pi/Y_{10} = Y_{2t}^\pi/Y_{20}$, $\forall t \in [0, T]$.*

Proposition 2.3 (equilibrium portfolio under index-based benchmarking) *Suppose that the parameters of the model are as in Proposition 2.2. Then the equilibrium portfolio under an index based benchmark portfolio for manager $i \in \{1, 2\}$, $i \neq j$, is given by the following expression*

$$\begin{aligned} \pi_{it}^{**} &= f_{i1t}^{**} \left[\frac{\gamma_i}{\gamma(\beta_i, \beta_j)} \pi_i^M + \frac{\phi\left(g_{i2}(\xi_t)\right) - \phi\left(g_{i1}(\xi_t)\right)}{\left[\Phi\left(g_{i1}(\xi_t)\right) - \Phi\left(g_{i2}(\xi_t)\right)\right]} h_t \right] \\ &+ f_{i2t}^{**} \left[\pi_i^M + \pi^Y \beta_i \left(1 - 1/\gamma_i\right) + \frac{\phi(g_3(\xi_t))}{\left[1 - \Phi\left(g_{i3}(\xi_t)\right)\right]} h_t \right] \\ &+ f_{i3t}^{**} \left[\pi^Y + \frac{-\phi\left(g_{i4}(\xi_t)\right)}{\Phi(g_{i4}(\xi_t))} h_t \right] + f_{i4t}^{**} \left[\pi^Y + \frac{\phi(g_{i5}(\xi_t))}{\left[1 - \Phi\left(g_{i5}(\xi_t)\right)\right]} h_t \right] \end{aligned}$$

where $\gamma(\beta_i, \beta_j)$, $\Phi(\cdot)$, $\phi(\cdot)$ and h_t are as defined in Proposition 2.2, and $\{\{f_{ikt}^{**}\}_{1 \leq k \leq 4}, \{g_{ik}\}_{1 \leq k \leq 4}\}$ are defined in the appendix.

Proposition 2.3 characterizes the optimal investment rule of fund manager i when the MRG restriction is index based (i.e., the solution to the problem of maximizing (2) subject to (8) and (7)). Similar to the previous case, the resulting equilibrium portfolio corresponds to a weighted sum of portfolio policies that are themselves optimal for different possible scenarios. In this case the scenarios correspond to the cases where neither, either i or j, or both managers are restricted by the benchmarking restriction at time T.

Compared to the peer-group benchmarking case, the index-based case has some particular features. In particular, the optimal portfolio in the scenario where manager i is the only one

unrestricted by the benchmarking constraint,

$$\pi_i^M + \pi^Y \beta_i \left(1 - 1/\gamma_i\right),$$

is a combination of two investment motives: one given by the adoption of the unconstrained optimal policy (π_i^M), and the other given by the optimal response to the strategy of his opponent (set equal to π^Y since the restriction is binding), where the latter depends on the strength of the relative performance bias (β_i) and the degree of relative risk aversion of manager (γ_i).

3   Discussion

Our results show that the combination of relative performance concerns and MRG affect asset allocation in important ways. First, MRG regulation alone gives rise to time varying investment policies, as the stochastic weights in π^* and π^{**} vary over time. This may be seen as openly favoring an active trading strategy as opposed to a passive one.

Second, when the benchmark portfolio is peer-group based, MRG favor the investment of the whole system to be driven by the least restricted manager (simply suppose that f_{i2t}^* is relatively bigger than f_{i3t}^*). Although this may not have been a conscious decision, it is relevant for policy making purposes to ask whether this is something that is indeed desired, specially when managers interest in relative performance may drive the asset allocation of the whole pension system to a suboptimal position.

Third, it is important to note that our model does not give any useful purpose for the MRG, and hence leaves open the motivation for its existence. In this sense, if MRG are thought of as a way to limit the risks of financial downturns, our results show that a MRG based on a peer-group benchmark may not be best way to achieve this, as it leaves the asset allocation completely driven by the risk preferences and relative performance concerns of asset managers. In fact, the only case where the institutional design makes perfect sense is in the case where it is socially desirable for asset managers to adopt myopic portfolios, since competition may help to achieve higher values of β's. We believe that the own goals of the pension system, and the increasing evidence of time-varying coefficient of the model discard this alternative. Overall, an index-based MRG seems to be preferred as it restrict investment policies in a clear way.

Fourth, one of the main drawbacks of MRG such as those in (3) and (7) is that they motivate herding among portfolio managers, as can be deduced from the equilibrium portfolio policies subject to both index-based and a peer-group based benchmark portfolios. In this context, it naturally emerges the question of the relevance of regulation and relative performance concerns as contributors to the observed herding behavior in DC pension systems. Our characterization of the equilibrium portfolio policy helps to shed light on the issue. In particular, unless $\gamma_1 = \gamma_2 = 1$, when the relative performance bias is nearly absent, the equilibrium portfolio should tend to a

linear combination of the unconstrained equilibrium portfolios, which if different from the myopic portfolio can provide a testable implication to take the model to the data.

A Appendix

A.1   On the existence and uniqueness of equilibrium

The Nash equilibrium corresponds to mutually consistent best responses, $(\hat{X}_{1T}, \hat{X}_{2T})$. In this regard we have the following result.

Proposition A.1 *Let the economy be as in section 2.1. Then, there are two candidate Nash equilibria:*

1. *One where both managers are unconstrained by the benchmarking restriction (3), given by (for $i, j \in \{1, 2\}$, $i \neq j$)*

$$X_{iT}^* = y_i^{-1/\Gamma \gamma_i} y_j^{-\beta_i(\gamma_i - 1)/\Gamma} \xi_T^{-(\beta_i(\gamma_i - 1) + \gamma_j)/\Gamma}, \qquad for \ \xi_T \in \Xi_{uu},$$

   where

$$
\begin{aligned}
\Gamma &= \gamma_1 \gamma_2 \left(1 - \beta_1 \beta_2\right) + \beta_1 \beta_2 \left(\gamma_1 + \gamma_2 - 1\right), \\
\Xi_{uu} &\equiv \{\xi_T : \xi_T^{\Psi_i} \leq A_i \quad and \quad \xi_T^{\Psi_j} \leq A_j\}, \\
\Psi_i &\equiv 1 - \left(\gamma_i \left(1 - \beta_i\right) + \beta_i\right) \left(\beta_j \left(\gamma_j - 1\right) + \gamma_i\right) / \Gamma, \\
A_i &\equiv K_i^{-\gamma_i} y_j^{(\gamma_i(1-\beta_i)+\beta_i)/\Gamma \gamma_j} y_i^{(\gamma_i(1-\beta_i)+\beta_i)\beta_j(\gamma_j-1)/\Gamma - 1};
\end{aligned}
$$

   and

2. *One where at least one of the managers is restricted by (3), given by*

$$
\begin{cases}
X_{uT}^* = \left(y_u \xi_T\right)^{-1/\bar{\gamma}_u} K_x^{\beta_u(\gamma_u - 1)/\bar{\gamma}_u} \\
X_{xT}^* = K_x^{\gamma_u/\bar{\gamma}_u} \left(y_u \xi_T\right)^{-1/\bar{\gamma}_u}
\end{cases}
, \quad for \ \xi_T \in \Xi_{ux},
$$

   where (u, x) stand for the 'unrestricted' and 'restricted' manager, respectively, and

$$
\begin{aligned}
\bar{\gamma}_u &= \gamma_u \left(1 - \beta_u\right) + \beta_u, \\
\Xi_{ux} &\equiv \{\xi_T : \xi_T^{1-\bar{\gamma}_x/\bar{\gamma}_u} \gtrless B_u^{(u,x)} \quad as \quad 1 - \bar{\gamma}_x/\bar{\gamma}_u \gtrless 0\}, \ B_u^{(u,x)} = y_x^{-1} y_u^{\bar{\gamma}_x/\bar{\gamma}_u} K_x^{\bar{\gamma}_x \beta_u(1-\gamma_u)/\bar{\gamma}_u - \gamma_x}.
\end{aligned}
$$

   In both cases, $y_i > 0$ and $y_j > 0$ are such that

$$\mathbb{E}_0 \left[\xi_T X_{iT}^* 1_{\{\xi_T \in \Xi_{uu}\}} + \xi_T X_{uT}^* 1_{\{\xi_T \in \Xi_{ux}\}} + \xi_T X_{xT}^* 1_{\{\xi_T \in \Xi_{xu}\}}\right] = x_i, \qquad \forall i, u, x \in \{1, 2\}, \ i \neq j.$$

Regarding the existence and uniqueness of equilibrium we have the following result, which parallels the discussion in BM.

Corollary A.1 *Suppose that the parameters of the model are such that* $1 - \bar{\gamma}_1 \left(\beta_2 \left(\gamma_2 - 1 \right) + \gamma_1 \right) / \Gamma < 0$, $1 - \bar{\gamma}_2 \left(\beta_1 \left(\gamma_1 - 1 \right) + \gamma_2 \right) / \Gamma > 0$, *and* $1 - \bar{\gamma}_2 / \bar{\gamma}_1 < 0$. *It then follows that:*

(i) *a unique Nash equilibrium exists if*

$$\{\xi_a \leq \xi_T \leq \xi_b\} \cup \{\xi_T < \xi_c\} \cup \{\xi_T > \xi_d\} = \mathbb{R}_{++}, \ and$$
$$\{\xi_a \leq \xi_T \leq \xi_b\} \cap \{\xi_T < \xi_c\} \cap \{\xi_T > \xi_d\} = \varnothing;$$

(ii) *multiple equilibria may exist if*

$$\{\xi_a \leq \xi_T \leq \xi_b\} \cup \{\xi_T < \xi_c\} \cup \{\xi_T > \xi_d\} = \mathbb{R}_{++}, \ and$$
$$\{\xi_a \leq \xi_T \leq \xi_b\} \cap \{\xi_T < \xi_c\} \cap \{\xi_T > \xi_d\} \neq \varnothing; \ and$$

(iii) *none equilibrium may exist if*

$$\{\xi_a \leq \xi_T \leq \xi_b\} \cup \{\xi_T < \xi_c\} \cup \{\xi_T > \xi_d\} \subset \mathbb{R}_{++}, \ or$$
$$\{\xi_a \leq \xi_T \leq \xi_b\} \cap \{\xi_T < \xi_c\} \cap \{\xi_T > \xi_d\} \neq \varnothing;$$

where

$$
\begin{aligned}
\xi_a &= K_1^{-\gamma_1/\Psi_1} y_2^{\bar{\gamma}_1/\Gamma\Psi_1\gamma_2} y_1^{\bar{\gamma}_1\beta_2(\gamma_2-1)/\Gamma\Psi_1 - \Psi_1} \\
\xi_b &= K_2^{-\gamma_2/\Psi_2} y_1^{\bar{\gamma}_2/\Gamma\Psi_2\gamma_1} y_2^{\bar{\gamma}_2\beta_1(\gamma_1-1)/\Gamma\Psi_2 - \Psi_2} \\
\xi_c &= y_2^{-\bar{\gamma}_1/(\bar{\gamma}_1-\bar{\gamma}_2)} y_1^{\bar{\gamma}_2/(\bar{\gamma}_1-\bar{\gamma}_2)} K_2^{(\bar{\gamma}_2\beta_1(1-\gamma_1)-\bar{\gamma}_1\gamma_2)/(\bar{\gamma}_1-\bar{\gamma}_2)} \\
\xi_d &= y_1^{-\bar{\gamma}_2/(\bar{\gamma}_2-\bar{\gamma}_1)} y_2^{\bar{\gamma}_1/(\bar{\gamma}_2-\bar{\gamma}_1)} K_1^{(\bar{\gamma}_1\beta_2(1-\gamma_2)-\bar{\gamma}_2\gamma_1)/(\bar{\gamma}_2-\bar{\gamma}_1)}
\end{aligned}
$$

with $y_i > 0$, $K_i = 1/\left(e^{\varepsilon_i} \left(1 + x_j/x_i \right) - 1 \right) > 0$, $\bar{\gamma}_i = \gamma_i \left(1 - \beta_i \right) + \beta_i$, $\Gamma = \gamma_1\gamma_2 \left(1 - \beta_1\beta_2 \right) + \beta_1\beta_2 \left(\gamma_1 + \gamma_2 - 1 \right)$, *and* $\Psi_i = 1 - \left(\gamma_i \left(1 - \beta_i \right) + \beta_i \right) \left(\beta_j \left(\gamma_j - 1 \right) + \gamma_i \right) / \Gamma$, *for* $i \in \{1, 2\}$.

A.2 Proofs

Proof of Proposition 2.1. First, notice that $v_i(\cdot, \cdot)$ can be written as

$$v_i = \frac{\left(X_{iT} X_{jT}^{-\beta_i} \right)^{1-\gamma_i}}{1 - \gamma_i}, \ \text{for } i, j \in \{1, 2\}, \ i \neq j.$$

Next, for the benchmarking constraint we have the following set equality: $\{R_{iT} \geq R_T e^{-\varepsilon_i}\} \equiv \{X_{iT} \geq K_i X_{jT}\}$, where $K_i = x_i/\left(e^{\varepsilon_i}\left(x_i + x_j\right) - x_i\right)$. Then, the FOC from manager's i portfolio problem are therefore given by $\partial v_i(\cdot,\cdot)/\partial X_{iT} = y_i \xi_T$, where $y_i > 0$ is the Lagrange multiplier attached to the static budget constraint. In the absence of the benchmarking restriction the Cournot best response function of manager i would have been simply $\hat{X}_{iT} = (y_i \xi_T)^{-1/\gamma_i} X_{jT}^{\beta_i(\gamma_i-1)/\gamma_i}$ as in BM. Instead, taking the benchmarking restriction into account we obtain

$$
\hat{X}_{iT} = \left\{
\begin{array}{ll}
(y_i \xi_T)^{-1/\gamma_i} X_{jT}^{\beta_i(\gamma_i-1)/\gamma_i} & \text{if} \quad \xi_T \leq y_i^{-1} K_i^{-\gamma_i} X_{jT}^{-\bar{\gamma}_i} \\
K_i X_{jT} & \text{if} \quad \xi_T > y_i^{-1} K_i^{-\gamma_i} X_{jT}^{-\bar{\gamma}_i}
\end{array}
\right.
$$

where $\bar{\gamma}_i = \gamma_i\left(1 - \beta_i\right) + \beta_i$. ∎

Proof of Corollary 2.2. Since $\varepsilon_1 = \varepsilon_2 = \infty$, we are back in the case studied by of BM, except that now (r, μ, σ) are not constant. Hence, the Cournot best response functions are given by $\hat{X}_{iT} = (y_i \xi_T)^{-1/\gamma_i} X_{jT}^{\beta_i(\gamma_i-1)/\gamma_i}$, for $i \in \{1,2\}$, $i \neq j$. Plugging $\hat{X}_{jT}$ into the expression for $\hat{X}_{jT}$ we obtain the (pure strategy Nash) equilibrium value for

$$
X_{iT}^* = y_i^{-\gamma_j/\Gamma} y_j^{\beta_i(1-\gamma_i)/\Gamma} \xi_T^{(\beta_i(1-\gamma_i)-\gamma_j)/\Gamma},
$$

where $\Gamma = \gamma_i \gamma_j \left(1 - \beta_i\beta_j\right) + \beta_i\beta_j\left(\gamma_i + \gamma_j - 1\right)$, and y_i and y_j are strictly positive constants that satisfy

$$
\mathbb{E}_0\left[\xi_T X_{iT}^*\right] = x_i \quad \text{and} \quad \mathbb{E}_0\left[\xi_T X_{jT}^*\right] = x_j.
$$

Finally, $\pi_{it}^* = \mathcal{D}_t(X_{it}^*)/X_{it}^* \sigma_t$ (e.g., Detemple et al. (2005)), where $\mathcal{D}_t(\cdot)$ is the Malliavin derivative operator,[7] and $X_{it}^* = \mathbb{E}_t\left[\xi_{t,T} X_{it}^*\right]$, from where we obtain

$$
\begin{aligned}
\pi_{it}^* &= \sigma_t^{-1}\theta_t + (\xi_t X_t^*)^{-1}\sigma_t^{-1}\mathbb{E}_t\left[\mathcal{D}_t\left(\xi_T X_{iT}^*\right)\right] \\
&= \frac{1}{\gamma(\beta_i,\beta_j)}\frac{\mu_t - r_t}{\sigma_t^2} + \frac{1}{\xi_t X_t^* \sigma_t}\mathbb{E}_t\left[\left(\frac{1}{\gamma(\beta_i,\beta_j)} - 1\right)H_{t,T}\xi_T X_{iT}^*\right]
\end{aligned}
$$

where $\gamma(\beta_i,\beta_j) \equiv \left(\gamma_i\gamma_j\left(1 - \beta_i\beta_j\right) + \beta_i\beta_j\left(\gamma_j + \gamma_i - 1\right)\right)/\left(\gamma_j + \beta_i\left(\gamma_i - 1\right)\right)$ and

$$
H_{t,T} \equiv \int_t^T (\mathcal{D}_t r_s + \theta_s \mathcal{D}_t \theta_s)ds + \int_t^T \mathcal{D}_t \theta_s dW_s.
$$

The result follows directly from the fact that $\gamma(1,1) = 1$. ∎

---

[7] The introduction of this operator in portfolio choice problems is due to Ocone and Karatzas (1991). The Malliavin derivative operator is an extension of the classical notion, that extends the concept to functions of the trajectories of W. In the same way that the classical derivative measures the local change in the function, due to a local change in the underlying variable, the Malliavin derivative measures the change in the function implied by a small change in the trajectory of W. The interested reader is referred to Detemple et al. (2003, 2005) for a brief introduction to this operator in the context of a portfolio choice problem and to Nualart (2006) for a comprehensive treatment.

Proof of Corollary 2.3. The assertion follows directly from Corollary 2.2 and the fact that $\gamma(0, \beta_j) = \gamma_i$. ∎

Proof of Proposition A.1. To find a Nash equilibrium we need to look for mutually consistent best responses for both managers. In our setting there are four possible combinations, depending on whether each manager is either restricted or unrestricted by the benchmarking constraint. The set of possible cases is hence given by the pairs (unrestricted, unrestricted), (unrestricted, restricted), (restricted, unrestricted), and (restricted, restricted), where each pair is to be read as (outcome for manager 1, outcome for manager 2). Since both managers cannot be restricted by the benchmarking constraint simultaneously we are left with only three possible cases.[8] For the (unrestricted, unrestricted) case it follows that the mutually consistent best responses are given by, for $i \in \{1, 2\}$, $i \neq j$:

$$
\begin{aligned}
X_{iT}^* &= (y_i \xi_T)^{-1/\gamma_i} \left(\hat{X}_{jT} \right)^{\beta_i (\gamma_i - 1)/\gamma_i} \\
&= y_i^{-\gamma_j/\Gamma} y_j^{\beta_i (1 - \gamma_i)/\Gamma} \xi_T^{(\beta_i (1 - \gamma_i) - \gamma_j)/\Gamma},
\end{aligned}
$$

where $\Gamma = \gamma_i \gamma_j \left(1 - \beta_i \beta_j \right) + \beta_i \beta_j \left(\gamma_i + \gamma_j - 1 \right)$, which holds in the interval

$$
\begin{aligned}
\xi_T^{\Psi_i} &\leq K_i^{-\gamma_i} y_j^{(\gamma_i (1 - \beta_i) + \beta_i)/\Gamma \gamma_j} y_i^{(\gamma_i (1 - \beta_i) + \beta_i)\beta_j (\gamma_j - 1)/\Gamma - 1}, \quad \text{and} \\
\xi_T^{\Psi_j} &\leq K_j^{-\gamma_j} y_i^{(\gamma_j (1 - \beta_j) + \beta_j)/\Gamma \gamma_i} y_j^{(\gamma_j (1 - \beta_j) + \beta_j)\beta_i (\gamma_i - 1)/\Gamma - 1}
\end{aligned}
$$

where

$$
\Psi_i = 1 - \left(\gamma_i \left(1 - \beta_i \right) + \beta_i \right) \left(\beta_j \left(\gamma_j - 1 \right) + \gamma_i \right) / \Gamma
$$

and correspond to the limiting values of the SDF for which $X_{iT}^* = K_i X_{jT}^*$ and $X_{jT}^* = K_j X_{iT}^*$. For the two remaining cases, it follows that the mutually consistent best responses are given by

$$
\begin{cases}
X_{uT}^* = (y_u \xi_T)^{-1/\bar{\beta}_u \gamma_u} K_x^{\beta_u (\gamma_u - 1)/\bar{\beta}_u \gamma_u} \\
X_{xT}^* = K_x^{1/\bar{\beta}_u} (y_u \xi_T)^{-1/\bar{\beta}_u \gamma_u}
\end{cases}, \quad \text{for } \xi_T \gtrless B_b^{(u,x)} \text{ as } 1 - \bar{\gamma}_x/\bar{\gamma}_u \gtrless 0,
$$

where 'u' and 'x' stand for the 'unrestricted' and 'restricted' manager, respectively, $\Xi_{ux} \equiv \{\xi_T : \xi_T^{1 - \bar{\gamma}_x/\bar{\gamma}_u} \gtrless B_u^{(u,x)}\}$, $B_u^{(u,x)} = y_x^{-1} y_u^{\bar{\gamma}_x/\bar{\gamma}_u} K_x^{\bar{\gamma}_x \beta_u (1 - \gamma_u)/\bar{\gamma}_u - \gamma_x}$. ∎

---

[8] In fact, the (restricted, restricted) case is algebraically possible, although it holds only when $X_{1T}^* = X_{2T}^* = \xi_T = 0$, which is a set of measure zero.

Proof of Proposition 2.2. The optimal wealth process is given by

$$
\begin{aligned}
X_{it}^* &= \mathbb{E}_t\left[\xi_{t,T}\left(X_{iT}^* 1_{\{\hat{\xi}_a \le \xi_T \le \check{\xi}_b\}} + X_{uT}^* 1_{\{\xi_T < \hat{\xi}_c\}} + X_{xT}^* 1_{\{\xi_T > \hat{\xi}_d\}}\right)\right] \\
&= \mathbb{E}_t\left[\xi_t^{-1}\left\{ y_i^{-1/\bar{\beta}\gamma_i} y_j^{\beta_i(1-\gamma_i)/\bar{\beta}\gamma_i\gamma_j} \xi_T^{1+(\beta_i(1-\gamma_i)-\gamma_j)/\bar{\beta}\gamma_i\gamma_j}\right\} 1_{\{\xi_T \in [\hat{\xi}_a, \check{\xi}_a]\}}\right] \\
&\quad + (y_i)^{-1/\bar{\beta}_i\gamma_i} K_j^{\beta_i(\gamma_i-1)/\bar{\beta}_i\gamma_i} \xi_t^{-1} \mathbb{E}_t\left[\left\{\xi_T^{1-1/\bar{\beta}_i\gamma_i}\right\} 1_{\{\hat{\xi}_a \le \xi_T \le \check{\xi}_b\}}\right] \\
&\quad + (y_j)^{-1/\bar{\beta}_j\gamma_j} K_i^{\gamma_i/\bar{\gamma}_j} \xi_t^{-1} \mathbb{E}_t\left[\left(\left\{\xi_T^{1-1/\bar{\gamma}_j}\right\} 1_{\{\xi_T > \hat{\xi}_d\}}\right)\right] \\
&= B_{i1t} + B_{i2t} + B_{i3t}.
\end{aligned}
$$

Following similar steps to those in the proof of Proposition 2.3, the expression in Proposition 2.2 obtains. ∎

Proof of Proposition 2.3. First, notice that the new portfolio choice problem of manager i can be written in a similar way as problem (P0), simply by replacing R_T with R_T^Y. The resulting best response function is hence given by (see the proof of Proposition 2.1)

$$
\check{X}_{iT} = \begin{cases} (y_i\xi_T)^{-1/\gamma_i} X_{jT}^{\beta_i(\gamma_i-1)/\gamma_i} & \text{if } \xi_T \le y_i^{-1} e^{\gamma_i\varepsilon_i} Y_{iT}^{-\gamma_i} X_{jT}^{\beta_i(\gamma_i-1)} \\ Y_{iT}e^{-\varepsilon_i} & \text{if } \xi_T > y_i^{-1} e^{\gamma_i\varepsilon_i} Y_{iT}^{-\gamma_i} X_{jT}^{\beta_i(\gamma_i-1)} \end{cases} \quad, \text{ for } i,j \in \{1,2\}, \, i \ne j.
$$

The set of candidate Nash equilibria are hence given by the pairs (unrestricted, unrestricted), (unrestricted, restricted), (restricted, unrestricted), and (restricted, restricted), where each pair is to be read as (outcome for manager 1, outcome for manager 2). Since the value of the benchmark portfolio is independent of the actual investment plan chosen by each manager, it is possible for the (restricted, restricted) case to emerge as a possible equilibrium.

For the (unconstrained, unconstrained) case we have that the equilibrium

$$
\begin{aligned}
X_{iT}^{**} &= (y_i\xi_T)^{-1/\gamma_i}\left((y_j\xi_T)^{-1/\gamma_j} (X_{iT}^{**})^{\beta_j(\gamma_j-1)/\gamma_j}\right)^{\beta_i(\gamma_i-1)/\gamma_i} \\
&= (y_i\xi_T)^{-1/\gamma_i}\left((y_j\xi_T)^{-\beta_i(\gamma_i-1)/\gamma_i\gamma_j}\right) (X_{iT}^{**})^{\beta_i\beta_j(\gamma_j-1)(\gamma_i-1)/\gamma_i\gamma_j} \\
&= (y_i\xi_T)^{-\gamma_j/\Gamma} (y_j\xi_T)^{-\beta_i(\gamma_i-1)/\Gamma}
\end{aligned}
$$

which holds in the interval

$$
\begin{aligned}
&\xi_T^{1+\beta_i(\gamma_i-1)\left(\beta_j(\gamma_j-1)+\gamma_i\right)/\Gamma+\gamma_i\sigma\psi/\theta} \\
&\le e^{\gamma_i\varepsilon_i} (Y_{i0}k_0)^{-\gamma_i} y_j^{-\gamma_i\beta_i(\gamma_i-1)/\Gamma} y_i^{-\beta_i(\gamma_i-1)\left(\beta_j(\gamma_j-1)+\gamma_i\right)/\Gamma-1}, \quad \text{and} \\
&\xi_T^{1+\beta_j(\gamma_j-1)\left(\beta_i(\gamma_i-1)+\gamma_j\right)/\Gamma+\gamma_j\sigma\psi/\theta} \\
&\le e^{\gamma_j\varepsilon_j} (Y_{j0}k_0)^{-\gamma_j} y_i^{-\gamma_j\beta_j(\gamma_j-1)/\Gamma} y_j^{-\beta_j(\gamma_j-1)\left(\beta_i(\gamma_i-1)+\gamma_j\right)/\Gamma-1}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
Y_{iT} &= Y_{i0} \exp\left\{ (r + \psi(\mu - r))T + \sigma\psi W_T \right\} \\
&= Y_{i0} \exp\left\{ (r + \psi(\mu - r))T \right\} \exp\left\{ -\theta W_T \right\}^{-\sigma\psi/\theta} \\
&= Y_{i0} \exp\left\{ (r + \psi\sigma\theta/2 - r\sigma\psi/\theta)T \right\} \exp\left\{ -\left(r + \theta^2/2\right)T - \theta W_T \right\}^{-\sigma\psi/\theta} \\
&= Y_{i0} k_0 \xi_T^{-\sigma\psi/\theta}.
\end{aligned}
$$

Similarly, for the (restricted, unrestricted) and (unrestricted, restricted) cases we have

$$
\begin{cases}
X_{uT}^{**} = (y_u \xi_T)^{-1/\gamma_u} (X_{xT}^{**})^{\beta_u(\gamma_u - 1)/\gamma_u} \\
X_{xT}^{**} = Y_{xT} e^{-\varepsilon_x}
\end{cases},
$$

which holds in the interval

$$
\begin{aligned}
&\xi_T^{1 - \gamma_u \sigma\psi/\theta + \beta_u(\gamma_u - 1)\sigma\psi/\theta} \\
&\leq \; y_u^{-1} e^{\gamma_u \varepsilon_u} (Y_{u0} k_0)^{-\gamma_u} \left(Y_{x0} k_0 e^{-\varepsilon_x} \right)^{\beta_u(\gamma_u - 1)}, \quad \text{and} \\
&\xi_T^{1 + (\beta_x(\gamma_x - 1)/\gamma_u)(1 + \sigma\psi\beta_u(\gamma_u - 1)/\theta) - \gamma_x \sigma\psi/\theta} \\
&> \; y_x^{-1} e^{\gamma_x \varepsilon_x} (Y_{x0} k_0)^{-\gamma_x} \left((y_u)^{-1/\gamma_u} \left(Y_{x0} k_0 e^{-\varepsilon_x} \right)^{\beta_u(\gamma_u - 1)/\gamma_u} \right)^{\beta_x(\gamma_x - 1)}.
\end{aligned}
$$

Finally, for the (restricted, restricted) case we have

$$
\hat{X}_{iT}^{**} = Y_{iT} e^{-\varepsilon_i} \quad \text{and} \quad \hat{X}_{jT}^{**} = Y_{jT} e^{-\varepsilon_j}
$$

which hold in the interval

$$
\begin{aligned}
&\xi_T^{1 - (\gamma_i(1 - \beta_i) + \beta_i)(\sigma\psi/\theta)} \\
&> \; y_i^{-1} e^{\gamma_i \varepsilon_i} (Y_{i0} k)^{-\gamma_i} \left(Y_{j0} k_0 \xi_T^{-\sigma\psi/\theta} e^{-\varepsilon_j} \right)^{\beta_i(\gamma_i - 1)}, \quad \text{and} \\
&\xi_T^{1 - (\gamma_j(1 - \beta_j) + \beta_j)(\sigma\psi/\theta)} \\
&> \; y_j^{-1} e^{\gamma_j \varepsilon_j} \left(Y_{j0} k_0 \xi_T^{-\sigma\psi/\theta} \right)^{-\gamma_j} \left(Y_{i0} k_0 e^{-\varepsilon_i} \right)^{\beta_j(\gamma_j - 1)}.
\end{aligned}
$$

The optimal wealth process is then given by

$$
X_{it}^{**} = \mathbb{E}_t \left[\xi_{t,T} \left(X_{iT}^{**} \mathbf{1}_{\{\xi_T \in \mathcal{Y}_{uu}\}} + X_{uT}^{**} \mathbf{1}_{\{\xi_T \in \mathcal{Y}_{ux}\}} + X_{xT}^{**} \mathbf{1}_{\{\xi_T \in \mathcal{Y}_{xu}\}} + \hat{X}_{iT}^{**} \mathbf{1}_{\{\xi_T \in \mathcal{Y}_{xx}\}} \right) \right]
$$

where $\mathcal{Y}_{uu}$ is deduced from the context.

Given the assumptions on the parameters, we have

$$
\begin{aligned}
X_{it}^{**} &= \mathbb{E}_t\left[\xi_{t,T}\left(X_{iT}^{**}1_{\{\xi_a\leq\xi_T\leq\xi_b\}} + X_{uT}^{**}1_{\{\xi_T\leq\xi_c\}} + X_{xT}^{**}1_{\{\xi_T<\xi_d\}} + \hat{X}_{iT}^{**}1_{\{\xi_T>\max(\xi_e,\xi_f)\}}\right)\right] \quad\text{(A.1)}\\
&\quad C_{1t} + \cdots + C_{4t}.
\end{aligned}
$$

The first term is given by

$$
\begin{aligned}
&\mathbb{E}_t\left[\xi_t^{-1}\left(y_i^{-\gamma_j/\Gamma}y_j^{-\beta_i(\gamma_i-1)/\Gamma}\xi_T^{1-(\gamma_j+\beta_i(\gamma_i-1))/\Gamma}\right)1_{\{\xi_a\leq\xi_T\leq\xi_b\}}\right]\\
&= y_i^{-\gamma_j/\Gamma}y_j^{-\beta_i(\gamma_i-1)/\Gamma}\xi_t^{A_{11}-1}e^{-A_{11}(r+\theta^2/2)(T-t)}\left[\int_{z\in\mathcal{Z}_1(\xi_t)}\frac{1}{\sqrt{2\pi}}e^{-A_{11}\theta z\sqrt{T-t}-z^2/2}dz\right]
\end{aligned}
$$

where

$$
A_{11} = 1 - 1/\gamma_u - (\sigma\psi/\theta)\beta_u(\gamma_u-1)/\gamma_u
$$

and

$$
\begin{aligned}
\{\xi_a \leq \xi_T \leq \xi_b\} &\Leftrightarrow \left\{\frac{\ln(\xi_t/\xi_b)-[r+\theta^2/2](T-t)}{\theta\sqrt{T-t}} \leq z \leq \frac{\ln(\xi_t/\xi_a)-[r+\theta^2/2](T-t)}{\theta\sqrt{T-t}}\right\}\\
&\Leftrightarrow \mathcal{Z}_1(\xi_t) \equiv \left\{z: \frac{\ln(\xi_t/\xi_b)-[r+\theta^2/2](T-t)}{\theta\sqrt{T-t}} \leq z \leq \frac{\ln(\xi_t/\xi_a)-[r+\theta^2/2](T-t)}{\theta\sqrt{T-t}}\right\}.
\end{aligned}
$$

The derivation of the other terms follows the same logic and its is left as an exercise for the interested reader.

Equilibrium portfolio policy then comes from $\pi_{it}^{**} = \mathcal{D}_t(X_{it}^{**})/X_{it}^{**}\sigma$. In addition, since each of the terms inside the expectation can be written as either $\Phi(g(\xi_t))$, $1-\Phi(g(\xi_t))$, or $\Phi(g(\xi_t)) - \Phi(g(\xi_t))$ it follows that

$$
\begin{aligned}
\mathcal{D}_t(X_{it}^{**}) &= C_{i1t}\left[(1-A_{11})\theta + \frac{\phi(g_{1i}(\xi_t))-\phi(g_a(\xi_t))}{\Phi(g_a(\xi_t))-\Phi(g_b(\xi_t))}h\right] + \cdots\\
&\quad + C_{i4t}\left[(1-A_{14})\theta + \frac{-\phi(g_e(\xi_t))}{\Phi(g_e(\xi_t))}h)\right]\\
\pi_{it}^{**} &= f_{i1t}^{**}\left[\frac{\gamma_i}{\gamma(\beta_i,\beta_j)}\pi_i^M + \frac{\phi(g_b(\xi_t))-\phi(g_a(\xi_t))}{\Phi(g_a(\xi_t))-\Phi(g_b(\xi_t))}h\right]\\
&\quad + \cdots + f_{i4t}^{**}\left[\pi^Y + \frac{-\phi(g_{ef}(\xi_t))}{\Phi(g_{ef}(\xi_t))}h(\xi_t)\right]
\end{aligned}
$$

where C_{it} stand for the optimal wealth of each of the four terms in (A.1). ∎

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